

# A GEOMETRIC INVARIANT OF 6-DIMENSIONAL SUBSPACES OF $4 \times 4$ MATRICES

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ABSTRACT. Let  $R$  be a 6-dimensional subspace of  $M_4(k)$ , the ring of  $4 \times 4$  matrices over an algebraically closed field  $k$ . We associate to  $R$  a closed subscheme  $\mathbf{X}_R$  of the Grassmannian of 2-dimensional subspaces of  $k^4$ . To define  $\mathbf{X}_R$  we write  $M_4(R)$  as  $V \otimes V^*$  where  $V$  is a 4-dimensional vector space. The reduced subscheme of  $\mathbf{X}_R$  is the set of 2-dimensional subspaces  $Q \subseteq V$  such that  $(Q \otimes V^*) \cap R \neq 0$ . Our main result is that if  $\dim(\mathbf{X}_R) = 1$ , the minimal possible dimension, then the degree of  $\mathbf{X}_R$  as a subscheme of the ambient Plücker  $\mathbb{P}^5$  is 20. We give two examples of  $\mathbf{X}_R$  that involve elliptic curves: in one case  $\mathbf{X}_R$  is a  $\mathbb{P}^1$ -bundle over an elliptic curve; in the other it is a curve having 7 irreducible components, three of which are elliptic curves, and four of which are smooth conics.

## 1. INTRODUCTION

We work over a fixed algebraically closed field  $k$

1.1. Let  $V$  be a 4-dimensional  $k$ -vector space. We identify the space of  $4 \times 4$  matrices over  $k$  with  $V \otimes V^*$ . Since we never make use to the multiplicative structure of the matrix algebra we will replace  $V^*$  by  $V$ .

We write  $G(2, V)$  for the Grassmannian of 2-planes in  $V$  and  $G(8, V^{\otimes 2})$  for the Grassmannian of 8-dimensional subspaces of  $V \otimes V$ . We will identify  $G(2, V)$  with its image in  $G(8, V^{\otimes 2})$  under the closed immersion  $G(2, V) \longrightarrow G(8, V^{\otimes 2})$ ,  $Q \mapsto Q \otimes V$ .

1.2. Let  $R$  be a 6-dimensional subspace of  $V \otimes V$ . The set

$$\mathbf{S}_R := \{W \in G(8, V^{\otimes 2}) \mid W \cap R \neq 0\}$$

is a closed subvariety of  $G(8, V^{\otimes 2})$ . It is a special Schubert variety [5, p.146]. The geometric invariant we associate to  $R$  is the scheme

$$\mathbf{X}_R := \mathbf{S}_R \cap G(2, V),$$

the scheme-theoretic intersection taken inside  $G(8, V^{\otimes 2})$ . Its reduced subscheme is

$$(\mathbf{X}_R)_{\text{red}} := \{Q \in G(2, V) \mid (Q \otimes V) \cap R \neq 0\} \subseteq G(2, V).$$

1.3. If  $R = \{x \otimes y - y \otimes x \mid x, y \in V\}$ , i.e., if  $R$  is the set of skew symmetric matrices, then  $\mathbf{X}_R = G(2, V)$ . More generally, if  $\theta \in \text{GL}(V)$  and  $R = \{x \otimes y - y \otimes \theta(x) \mid x, y \in V\}$ , then  $\mathbf{X}_R = G(2, V)$ .

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1.4. Our main result is

**Theorem 1.1** (Theorem 2.6 and Corollary 2.7). *For all  $R$ , all irreducible components of  $\mathbf{X}_R$  have dimension  $\geq 1$ , and the  $\mathbf{X}_R$ 's form a flat family over the locus of those  $R$  where  $\dim(\mathbf{X}_R) = 1$ .*

*Moreover, if  $\dim(\mathbf{X}_R) = 1$ , then  $\deg(\mathbf{X}_R) = 20$ , where its degree is computed by using the Plücker embedding  $\mathbf{G}(2, V) \rightarrow \mathbb{P}^5$  and viewing  $\mathbf{X}_R$  as a subscheme of  $\mathbb{P}^5$ .*

1.5. Our reading (ignorance?) of the linear algebra literature suggests that  $\mathbf{X}_R$  is a new invariant. Sometimes,  $\mathbf{X}_R$  is isomorphic to the subscheme of  $\mathbb{P}(R) \subseteq \mathbb{P}(V^{\otimes 2})$  consisting of the rank-2 tensors that belong to  $R$ ; i.e., the locally closed subscheme of  $\mathbb{P}(R)$  where all  $3 \times 3$  minors vanish.

Examples of that form prompted us to write this paper. Such examples appear in looking for non-commutative analogues of  $\mathbb{P}^3$  or, almost equivalently, from non-commutative analogues of the polynomial ring in four variables. We say more about our motivation in §4.

## 2. THE PROOF OF THEOREM 1.1

2.1. We will examine various families over the base scheme  $\mathbf{G}(6, V^{\otimes 2})$ . Define

$$\begin{aligned} \mathbf{G} &:= \mathbf{G}(6, V^{\otimes 2}) \times \mathbf{G}(8, V^{\otimes 2}), \\ \mathbf{G}(1, 3) &:= \mathbf{G}(6, V^{\otimes 2}) \times \mathbf{G}(2, V), \\ \mathbf{S}_R &:= \{W \in \mathbf{G}(8, V^{\otimes 2}) \mid W \cap R \neq \{0\}\}, \\ \mathbf{S} &:= \{(R, x) \mid R \in \mathbf{G}(6, V^{\otimes 2}), x \in \mathbf{X}_R\} \subseteq \mathbf{G}(1, 3), \\ \mathbf{X}_R &:= \mathbf{S}_R \cap \mathbf{G}(2, V), \quad \text{the scheme-theoretic intersection taken inside } \mathbf{G}(8, V^{\otimes 2}), \\ \mathbf{X} &:= \mathbf{S} \cap \mathbf{G}(1, 3), \quad \text{the scheme-theoretic intersection taken inside } \mathbf{G}(1, 3), \\ &= \{(R, x) \mid R \in \mathbf{G}(6, V^{\otimes 2}), x \in \mathbf{X}_R\} \subseteq \mathbf{G}(1, 3), \\ U &:= \{R \in \mathbf{G}(6, V^{\otimes 2}) \mid \dim(\mathbf{X}_R) = 1\} \subseteq \mathbf{G}(6, V^{\otimes 2}), \\ \mathbf{X}_U &:= \{(R, x) \mid \dim(\mathbf{X}_R) = 1, x \in \mathbf{X}_R\} \subseteq \mathbf{X}. \end{aligned}$$

and

$$\begin{aligned} \pi : \mathbf{G} &\longrightarrow \mathbf{G}(6, V^{\otimes 2}), \quad \text{the projection,} \\ f : \mathbf{X} &\longrightarrow \mathbf{G}(6, V^{\otimes 2}), \quad \text{the restriction of } \pi \text{ to } \mathbf{X}. \end{aligned}$$

Thus  $\mathbf{X}_U = f^{-1}(U) = \pi^{-1}(U) \cap \mathbf{X}$ . In general, we denote the restriction of a family to a subscheme  $U \subseteq \mathbf{G}(6, V^{\otimes 2})$  by the subscript  $U$ .

**Proposition 2.1.** *The set  $U$  is a dense open subset of  $\mathbf{G}(6, V^{\otimes 2})$ .*

*Proof.* Each  $\mathbf{S}_R$  is a Schubert variety and, by [6, pp. 193-196], for example, its codimension in  $\mathbf{G}(8, V^{\otimes 2})$  is 3. It follows that the codimension of  $\mathbf{S}$  in  $\mathbf{G}$  is 3. Since  $\dim(\mathbf{G}(2, V)) = 4$ , the codimension of  $\mathbf{G}(1, 3)$  in  $\mathbf{G}$  is  $\dim \mathbf{G}(8, V^{\otimes 2}) - 4$ . It follows that every irreducible component  $\mathbf{X}_i$ ,  $i \in I$ , of  $\mathbf{S} \cap \mathbf{G}(1, 3) = \mathbf{X}$  has codimension  $\leq \dim \mathbf{G}(8, V^{\otimes 2}) - 1$ . In other words, the relative dimension,  $\dim \mathbf{X}_i - \dim \mathbf{G}(6, V^{\otimes 2})$ , is at least 1 for each  $i \in I$ .

We will now apply [7, Exercise II.3.22(d)] to the families obtained by restricting the projection  $\pi$  to the reduced subschemes  $(\mathbf{X}_i)_{\text{red}}$  (and corestricting to the scheme-theoretic images of these restrictions). The conclusion of the cited exercise is that  $\mathbf{Y} := \{x \in \mathbf{X} \mid \dim(\mathbf{X}_{\pi(x)}) \geq$

$2\}$  is a closed subscheme of  $\mathbf{X}$ . Since  $\pi|_{\mathbf{X}}$  is a projective morphism it is closed. Since  $\mathbf{X}_U = \mathbf{G}(6, V^{\otimes 2}) - \pi(\mathbf{Y})$ ,  $U$  is open. By [15],  $U$  is non-empty, and therefore dense.  $\blacksquare$

The following result is embedded in the proof of [Proposition 2.1](#).

**Proposition 2.2.** *Let  $R \in \mathbf{G}(6, V^{\otimes 2})$ . If  $\dim(\mathbf{X}_R) = 1$ , then every irreducible component of  $\mathbf{X}_R$  has dimension 1.*

**Proposition 2.3.** *Let  $G$  be an algebraic group acting on  $\mathbb{P}^n$ . Let  $T$  be an integral noetherian scheme endowed with a transitive action of  $G$ . Let  $G$  act diagonally on  $T \times \mathbb{P}^n = \mathbb{P}_T^n$ . If  $\mathbf{X} \subseteq \mathbb{P}_T^n$  a  $G$ -stable closed subscheme, then  $\mathbf{X}$  is flat over  $T$ .*

*Proof.* Let  $t, t' \in T$ . There is  $g \in G$  such that  $t' = g(t)$ . The action of  $g$  is such that the diagram

$$\begin{array}{ccc} \mathbf{X}_t & \longrightarrow & \mathbb{P}_{k(t)}^n \\ g \downarrow & & \downarrow g \\ \mathbf{X}_{t'} & \longrightarrow & \mathbb{P}_{k(t')}^n \end{array}$$

commutes. It follows that  $\mathbf{X}_t$  and  $\mathbf{X}_{t'}$  have the same Hilbert polynomial. It now follows from [7, Thm. III.9.9] that  $\mathbf{X}$  is flat over  $T$ .  $\blacksquare$

**Corollary 2.4.**  *$\mathbf{S}$  is flat over  $\mathbf{G}(6, V^{\otimes 2})$ .*

*Proof.* Let  $\mathbf{G}(8, V^{\otimes 2}) \rightarrow \mathbb{P}(\wedge^8(V^{\otimes 2}))$  be the Plücker embedding and consider  $\mathbf{S}$  as a closed subscheme of  $\mathbf{G}(6, V^{\otimes 2}) \times \mathbb{P}(\wedge^8(V^{\otimes 2}))$ . Let  $\mathrm{GL}(V^{\otimes 2})$  act diagonally on the previous product in the obvious way. It is clear that the closed subscheme  $\mathbf{S}$  is stable under the action of  $\mathrm{GL}(V^{\otimes 2})$  and that the action of  $\mathrm{GL}(V^{\otimes 2})$  on  $\mathbf{G}(6, V^{\otimes 2})$  is transitive. The result now follows from [Proposition 2.3](#).  $\blacksquare$

**Lemma 2.5.** *Both  $\mathbf{X}_U$  and  $\mathbf{S}$  are Cohen-Macaulay schemes.*

*Proof.* Let  $\pi : \mathbf{S} \rightarrow \mathbf{G}(6, V^{\otimes 2})$  be the restriction of the projection. Hochster proved that Schubert varieties in Grassmannians are Cohen-Macaulay [8]. Thus,  $\mathbf{S}_R$  is Cohen-Macaulay for all  $R \in \mathbf{G}(6, V^{\otimes 2})$ . Let  $x \in \mathbf{S}$ . By [Corollary 2.4](#), the natural map  $u : \mathcal{O}_{\pi(x), \mathbf{G}(6, V^{\otimes 2})} \rightarrow \mathcal{O}_{x, \mathbf{S}}$  is a flat homomorphism of noetherian local rings. The closed fibre of  $u$ , which is  $\mathcal{O}_{x, \mathbf{S}_{\pi(x)}}$ , is Cohen-Macaulay. Since  $\mathcal{O}_{\pi(x), \mathbf{G}(6, V^{\otimes 2})}$  is also Cohen-Macaulay, [10, Cor 23.3] implies that  $\mathcal{O}_{x, \mathbf{S}}$  is Cohen-Macaulay. Thus  $\mathbf{S}$  is Cohen-Macaulay.

The Cohen-Macaulay property for  $\mathbf{X}_U$  will follow from [4, Prop. 18.13] applied to the following setup.

Let  $A = \mathcal{O}_{x, \mathbf{X}_U}$  be the local ring at a point  $x \in \mathbf{X}_U$ . Recall that  $\mathbf{X}_U$  is the intersection  $\mathbf{S}_U \cap \mathbf{G}(1, 3)_U$ . Since  $\mathbf{G}(1, 3)_U$  is regular and hence a local complete intersection, we can find  $n = \dim \mathbf{G}(8, V^{\otimes 2}) - 4$  generators for the ideal  $I$  of  $A$  such that  $\mathcal{O}_{x, \mathbf{S}_U} = A/I$ .

Since we are restricting our families to  $U$ , the codimension of  $I$  in  $A$  is precisely  $n$ . But then the hypotheses of [4, Prop. 18.13] are met, and the ring  $A/I$  is Cohen-Macaulay.  $\blacksquare$

**Theorem 2.6.** *The scheme  $\mathbf{X}_U$  is flat over  $U$ .*

*Proof.* Let  $x \in \mathbf{X}_U = f^{-1}(U)$ . Let  $B = \mathcal{O}_{f(x), U}$  and  $A = \mathcal{O}_{x, \mathbf{X}_U}$ . We must show that  $A$  is a flat  $B$ -module. Let  $\mathfrak{p}$  be the maximal ideal in  $B$ .

Because  $\mathbf{X}_U$  is Cohen-Macaulay,  $A$  is a Cohen-Macaulay ring. Since  $\dim((\mathbf{X}_U)_{\pi(x)}) = 1$ ,  $\mathrm{Kdim}(A) = \mathrm{Kdim}(B) + \mathrm{Kdim}(A/A\mathfrak{p})$ . Taken together, this equality, the fact that  $A$  is Cohen-Macaulay, and [4, Thm. 18.16(b)], imply that  $A$  is a flat  $B$ -module.  $\blacksquare$

**Corollary 2.7.** *If  $\dim(\mathbf{X}_R) = 1$ , then  $\deg(\mathbf{X}_R) = 20$  where the degree is taken inside the ambient “Plücker projective space”  $\mathbb{P}^5 \supseteq \mathbb{G}(1, 3)$ .*

*Proof.* Since  $\mathbf{X}_U$  is flat over  $U$  and  $k$  is algebraically closed, the Hilbert polynomial of  $X_R$ , viewed as a closed subscheme of  $\mathbb{P}^5$ , is the same for all  $R \in U$  [7, Thm. III.9.9]. Since  $\dim(\mathbf{X}_R) = 1$ , the leading coefficient of its Hilbert polynomial is the degree of  $\mathbf{X}_R$  as a closed subscheme of  $\mathbb{P}^5$ . Thus,  $\deg(\mathbf{X}_R)$  is the same for all  $b \in U$ . We therefore need only exhibit a single  $R$  for which  $\deg(\mathbf{X}_R) = 20$ . This is done in [1, Theorem 3.3].  $\blacksquare$

This completes the proof of Theorem 1.1.

### 3. EXAMPLES

Throughout this section suppose the characteristic of  $k$  is not 2.

**3.1. An example with  $\mathbf{X}_R$  a surface.** Let  $\{\alpha_1, \alpha_2, \alpha_3\} \subseteq k - \{0, \pm 1\}$  and suppose that  $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3 = 0$ . Let  $\{x_0, x_1, x_2, x_3\}$  be a basis for a 4-dimensional vector space  $V$ . Let  $E$  be the quartic elliptic curve in  $\mathbb{P}^3$  given by the intersection of any two of the quadrics

$$\begin{cases} x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0, \\ x_0^2 - \alpha_2\alpha_3x_1^2 - \alpha_3x_2^2 + \alpha_2x_3^2 = 0, \\ x_0^2 + \alpha_3x_1^2 - \alpha_1\alpha_3x_2^2 - \alpha_1x_3^2 = 0, \\ x_0^2 - \alpha_2x_1^2 + \alpha_1x_2^2 - \alpha_1\alpha_2x_3^2 = 0. \end{cases} \quad (3.1)$$

Every elliptic curve is isomorphic to at least one of these  $E$ 's.

To save space we write  $x_ix_j$  for  $x_i \otimes x_j \in V^{\otimes 2}$ . Let  $R$  be the linear span of the 6 elements

$$x_0x_i - x_ix_0 - \alpha_i(x_jx_k + x_kx_j) \quad \text{and} \quad x_0x_i + x_ix_0 - \alpha_i(x_jx_k - x_kx_j) \quad (3.2)$$

where  $(i, j, k)$  runs over the cyclic permutations of  $(1, 2, 3)$ .

By [9] and [20],  $\mathbf{X}_R$  is isomorphic to a  $\mathbb{P}^1$ -bundle over  $E$ . More precisely,  $\mathbf{X}_R$  consists of the lines in  $\mathbb{P}^3$  whose scheme-theoretic intersection with  $E$  has multiplicity  $\geq 2$ , and hence  $= 2$ . In other words,  $\mathbf{X}_R$  parametrizes the set of secant lines to  $E$ .

**3.2. An example with  $\dim(\mathbf{X}_R) = 1$ .** Retain the notation in §3.1, and let  $R'$  be the linear span of the 6 elements

$$x_0x_i - x_ix_0 - \alpha_i(x_jx_k - x_kx_j) \quad \text{and} \quad x_0x_i + x_ix_0 - \alpha_i(x_jx_k + x_kx_j) \quad (3.3)$$

where  $(i, j, k)$  is a cyclic permutation of  $(1, 2, 3)$ . By [3],

$$\mathbf{X}_{R'} = (C_1 \sqcup C_2 \sqcup C_3 \sqcup C_4) \cup (E_1 \sqcup E_2 \sqcup E_3),$$

where

- (1) the  $C_i$ 's are disjoint smooth plane conics,
- (2) the  $E_j$ 's are degree-4 elliptic curves that span different 3-planes in  $\mathbb{P}^5$ , and
- (3) each  $E_j$  is isomorphic to  $E/(\xi_j)$  where  $(\xi_1), (\xi_2), (\xi_3)$  are the three order-2 subgroups of  $E$ , and
- (4)  $|C_i \cap E_j| = 2$  for all  $(i, j) \in \{1, 2, 3, 4\} \times \{1, 2, 3\}$ .

**3.3. The origin of the examples in §§3.1 and 3.2.** There is a point  $\tau \in E$  such that the space  $R$  given by (3.2) has the following description (see [20] for example).

Let  $\mathcal{L}$  be a line bundle of degree four on  $E$ , and let  $V = H^0(E, \mathcal{L})$ . Then  $R$  consists of those sections of the line bundle  $\mathcal{L} \boxtimes \mathcal{L}$  on  $E \times E$  whose divisor of zeros is of the form

$$D + \{(x, x + 2\tau) \mid x \in E\},$$

where  $D$  is a divisor invariant under the involution  $(x, y) \mapsto (y + 2\tau, x - 2\tau)$  such that

$$\{(x, x - 2\tau)\}$$

(the fixed-point set of the involution) occurs in  $D$  with even multiplicity.

The quotient  $TV/(R)$  of the tensor algebra on  $V$  modulo the ideal generated by  $R$  is a characteristic-free construction of the four-dimensional Sklyanin algebras of [16, 17, 11] introduced by Sklyanin to study certain solutions to the Yang-Baxter equation. By [18], the algebras  $TV/(R)$  form a flat family of deformations of the symmetric algebra  $SV$ , the polynomial ring on four variables, with the data  $(E, \tau)$  acting as deformation parameters.

The algebras  $TV/(R')$  can be obtained from the Sklyanin algebras  $TV/(R)$  by means of a cocycle twist construction, so they too have a geometric construction involving an elliptic curve. We refer to [2, 3] for details and the description of  $\mathbf{X}_{R'}$ .

## 4. MOTIVATION

**4.1. Non-commutative algebraic geometry.** This paper grew out of our interest in non-commutative analogues of  $\mathbb{P}^3$ .

Let  $V$  be a 4-dimensional vector space. Then  $\mathbb{P}^3 = \text{Proj}(SV)$ , the projective scheme whose homogeneous coordinate ring is the polynomial ring on four variables. In [12], Serre proved that a certain quotient of the category of graded  $SV$ -modules is equivalent to  $\text{Qcoh}(\mathbb{P}^3)$ , the category of quasi-coherent sheaves on  $\mathbb{P}^3$ .

Since  $SV = TV/(\text{Alt})$  where  $\text{Alt}$  is the subspace of  $V^{\otimes 2}$  consisting of skew-symmetric tensors one might perform the same quotient category construction on other algebras  $A = TV/(R)$  when  $R$  is any 6-dimensional subspace  $V^{\otimes 2}$  and, if  $A$  is “good”, one might hope that this quotient category behaves “like”  $\text{Qcoh}(\mathbb{P}^3)$ . This hope is a reality in a surprising number of cases and has led to a rich subject that goes by the name of non-commutative algebraic geometry—see [19] and the references therein.

From now on we will assume that  $A$  is “good” without saying what that means.

We will denote the appropriate quotient of the category of graded  $A$ -modules by  $\text{QGr}(A)$ . We will write  $\text{Proj}_{nc}(A)$  for the “non-commutative analogue of  $\mathbb{P}^3$ ” that has  $A$  as its homogeneous coordinate ring. This is a fictional object that is made manifest by declaring the category of quasi-coherent sheaves on  $\text{Proj}_{nc}(A)$  to be  $\text{QGr}(A)$ .

**4.2. Points, lines, etc.** The most elementary geometric features of  $\mathbb{P}^3$  are points, lines, planes, quadrics, and their incidence relations. There are non-commutative analogues of these and a first investigation of  $\text{Proj}_{nc}(A)$  involves finding its “points” and “lines” and the “incidence relations” among them.

A non-commutative ring usually has far fewer two-sided ideals than a commutative ring. Even when  $A$  is “good” it has far fewer two sided ideals than  $SV$ . As a consequence  $\text{Proj}_{nc}(A)$  usually has far fewer “points” and “lines” than does  $\mathbb{P}^3$ .

For example,  $\text{Proj}_{nc}(A)$  can have as few as 20 “points”, which is the case for  $TV/(R')$  when  $R'$  is the space in §3.2 [2], or even just one point with multiplicity 20 [13]. The “points”

in  $\text{Proj}_{nc}(A)$  form a scheme that is called the **point scheme**. When  $A$  is “good”, that point scheme is a closed subscheme of  $\mathbb{P}^3$ . When the point scheme has dimension zero its degree is 20.

By [14, 15], there is also a **line scheme** that classifies the “lines” in  $\text{Proj}_{nc}(A)$ . By [14], the line scheme is isomorphic to  $\mathbf{X}_R \subseteq \mathbf{G}(2, V)$  and always has dimension  $\geq 1$ . Thus, the main result in this paper says that if  $A$  is “good” and the line scheme has dimension 1, then its degree is 20.

**4.3. Rank-2 elements in  $R$ .** As before,  $R$  denotes a 6-dimensional subspace of  $V \otimes V$ .

The rank of an element  $t \in V^{\otimes 2}$  is the smallest  $n$  such that  $t = v_1 \otimes w_1 + \cdots + v_n \otimes w_n$  for some  $v_i, w_i \in V$ . The elements in  $V^{\otimes 2}$  of rank  $\leq r$  form a closed subvariety of  $V^{\otimes 2}$ . This subvariety is a union of 1-dimensional subspaces so the lines through 0 and the non-zero elements of rank  $\leq r$  form a closed subvariety  $T_{\leq r}$  of  $\mathbb{P}(V^{\otimes 2})$ . We write  $T_r$  for  $T_{\leq r} - T_{\leq r-1}$ . Define  $\mathbf{Z}_R$  to be the scheme-theoretic intersection

$$\mathbf{Z}_R := T_2 \cap \mathbb{P}(R).$$

Define  $\mathbf{Y}_R$  to be the scheme-theoretic intersection

$$\mathbf{Y}_R := \mathbf{S}_R \cap \mathbf{G}'(2, V)$$

in  $\mathbf{G}(8, V^{\otimes 2})$  where  $\mathbf{G}'(2, V)$  is the image of the closed immersion of  $\mathbf{G}(2, V)$  in  $\mathbf{G}(8, V^{\otimes 2})$  given by  $Q \mapsto V \otimes Q$ . Thus,  $\mathbf{Y}_R = \mathbf{X}_{\sigma(R)}$  where  $\sigma : V^{\otimes 2} \rightarrow V^{\otimes 2}$  is the linear map  $\sigma(u \otimes v) = v \otimes u$ .

If  $t \in \mathbf{Z}_R$  and  $t = a \otimes b + c \otimes d$ , then the 2-dimensional subspace of  $V$  spanned by  $a$  and  $b$  depends only on  $t$  and not on its representation as  $a \otimes b + c \otimes d$ . Thus, there is a well-defined morphism  $\phi : \mathbf{Z}_R \rightarrow \mathbf{G}(2, V)$ ,  $\phi(a \otimes b + c \otimes d) := (ka + kc)$ . The image of  $\phi$  is  $\mathbf{X}_R$ . There is a similar morphism  $\mathbf{Z}_R \rightarrow \mathbf{G}(2, V)$ ,  $(a \otimes b + c \otimes d) \mapsto (kb + kd)$ . Thus, we have morphisms

$$\begin{array}{ccc} & \mathbf{Z}_R & \\ \phi \swarrow & & \searrow \\ \mathbf{X}_R & & \mathbf{Y}_R \end{array}$$

When  $TV/(R)$  is “good”, the morphism  $\phi$  is an isomorphism: our  $\phi$  is the same as the morphism  $\phi$  in [14, Lemma 2.5] where it is proved to be an isomorphism (our notation differs from that in loc. cit.).

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